How Landau-Zener Tunneling Takes Time

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We calculate the time evolution of Landau-Zener tunneling for atoms in an accelerating optical lattice. Analytical expressions are obtained that are in good agreement with a recent observation of nonexponential decay. We identify new experimental regimes that show a crossover from strong coherent oscillations to exponential decay in the temporal evolution of the survival probability. We establish the time scale of this crossover, and make connection to a tunneling time of Zener breakdown. [S0031-9007(98)05754-8]

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Unstable quantum systems exhibit exponential decay of the survival probability, but deviations from this law at short and long times were predicted [1,2]. The experimental observation has not been possible [3] until recently, when clear evidence of short-time deviation from exponential decay was seen in the survival probability of ultracold atoms tracking an accelerating optical lattice [4].

In this Letter, we provide a theoretical analysis of the atomic-optical system in terms of Landau-Zener (LZ) tunneling between Bloch bands [5]. We find an initial nonexponential regime that starts with a quadratic time dependence, then becomes a damped oscillation, and finally settles into exponential decay. We establish the characteristic time separating the two regimes, and make connection to a tunneling time of Zener breakdown [6,7]. We suggest future experiments for the measurement of this time.

We consider an ensemble of ultracold atoms in an accelerating optical potential of the form $V_0 \cos(2k_L x - k_L a t^2)$, where a is the acceleration, k_L is the wave number of the laser that creates the potential, and V_0 is proportional to the average light intensity. This system was used earlier to observe Bloch oscillations, Wannier-Stark ladders, and LZ tunneling [8]. In the accelerating reference frame, the Hamiltonian of the system may be written as

$$\frac{1}{2m}(p + mat)^2 + V_0 \cos(2k_L x), \qquad (1)$$

where the inertial force $-ma$ has been represented in the vector potential gauge. For simple notation, we take the system's natural units in which $\hbar = m = 2k_L = 1$.

The energy spectrum of the above Hamiltonian with zero acceleration is well known, and consists of Bloch bands separated by gaps [9]. As an initial condition, we assume that the lowest band is uniformly occupied, while the higher bands are empty. When an acceleration is imposed, the atomic quasimomentum changes as the atoms undergo Bloch oscillations. The atoms can also escape from the accelerating lattice by interband Landau-Zener tunneling and hence form an unstable quantum system. The tunneling rate in this case is given by $\Gamma = a \exp(-a_c/a)$, where a_c is a critical acceleration defined below [10]. Our goal is to calculate the survival probability in the lowest band as a function of time.

Model and formulation.—We will work in a regime where the time dependence due to acceleration is almost adiabatic for the first gap, but is sudden for the higher gaps. This is possible for a relatively weak potential, because for the first gap, $\epsilon_g = V_0$, while for the second gap, $\epsilon'_g = V_0^2$. Subsequent gaps scale with even higher orders of the potential. The critical accelerations for the first and second gaps are $a_c = \pi\epsilon_g^2/2$ and $a'_c = \pi(\epsilon'_g)^2/4$, respectively [11]. In the parameter space of a and V_0 (see Fig. 1), our theory will be valid inside the region bounded by the two curves $a = \frac{\pi}{2}V_0^2$ and $a = \frac{\pi}{4}V_0^4$, and to the left of the line $V_0 = 1$ [12]. Our band structure is modeled as in Fig. 2, where all of the gaps except for the first are neglected.

We start by expanding the wave function in terms of the eigenstates $e^{ikx}u_n(x, k + at)$ of the Hamiltonian (1) with energy $\epsilon_n(k + at)$ for a particular wave number k. To leading order in the nonadiabatic coupling $\langle \dot{u}_0 | u_n \rangle$, the

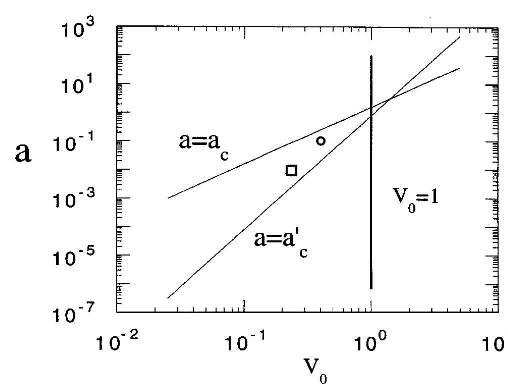

FIG. 1. The parameter space of potential V_0 and acceleration a in the system's natural units. The present theory is valid inside the region defined by $V_0 < 1$, $a'_c < a < a_c$. The circle corresponds to the recent experiment (Ref. [4]), and the square indicates a regime where coherent oscillations of the survival probability are predicted.

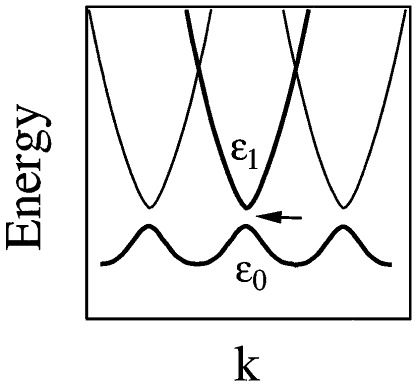

FIG. 2. Model band structure used in the theory, in which all of the gaps above the lowest gap are neglected. The arrow indicates the crossing point where tunneling occurs most easily.

survival amplitude C_0 in the lowest band changes at the rate

$$-\sum_{n\neq 0}\int_0^t dt'\, C_0(t')\,\langle u_n|\dot{u}_0\rangle_{t'}\langle\dot{u}_0|u_n\rangle_t\, e^{i\int_{t'}^{t}(\epsilon_0-\epsilon_n)dt''}, \quad (2)$$

where the subscripts t' and t indicate the times at which the corresponding quantities are evaluated. Because $C_0(t)$ changes slowly in time due to the smallness of the above expression, we can move $C_0(t')$ out of the integral by setting its argument to t. This replacement yields an exponent of the probability P_0 correct to leading order in the perturbation, which changes at a rate equal to the real part of the following expression:

$$-2\sum_{n\neq 0}\int_0^t dt'\langle u_n|\dot{u}_0\rangle_{t'}\langle\dot{u}_0|u_n\rangle_t\, e^{i\int_{t'}^{t}(\epsilon_0-\epsilon_n)dt''}. \quad (3)$$

Next, we make an average over the initial distribution in the lowest band. This amounts to integrating the above result over a Brillouin zone of size $2k_L = 1$ (in the system's natural units). The result is simplified by the following considerations. First, the integrand is periodic in k, and the time and k dependences enter only through the combinations $k + at$ and $k + at'$. Second, the set of the higher bands in a single Brillouin zone is equivalent to a single band, displayed as a thick curve among the higher bands in Fig. 2, over the entire k space [13]. Therefore, to leading order in the nonadiabatic perturbation, the exponent of the survival probability is of the form

$$\ln P_0 = -\int_0^t dt'(t - t')W(t'). \quad (4)$$

Here, the transition kernel $W(t)$ is equal to the real part of the following expression:

$$2a^2\int_{-\infty}^{\infty} dk\langle u_1|u_0'\rangle_{k-s}\langle u_0'|u_1\rangle_{k+s}\, e^{-i\int_{k-s}^{k+s}\epsilon_{10}dk'/a}, \quad (5)$$

where $s = at/2$, and $\epsilon_{10} = \epsilon_1 - \epsilon_0$ with ϵ_1 representing the single upper band in the extended k space.

Characteristics of time evolution.—At short times, the exponent (4) goes as $-\frac{1}{2}W(0)t^2$. This quadratic time dependence is a general property of quantum mechanics. Because the Schrödinger equation is first order in the time derivative, the transition amplitude out of an initial state must be linear in time at short times, implying a quadratic time dependence of the transition probability. At long times, the kernel $W(t)$ drops rapidly (see below) which leads to an exponential law for the decay of the survival probability. Replacing the upper limit in Eq. (4) by infinity, we find the asymptotic form of the survival probability as $P_0 = e^{-\Gamma t+\Gamma_0}$, with $\Gamma = \int_0^\infty W(t)\,dt$, and $\Gamma_0 = \int_0^\infty tW(t)\,dt$.

To show that the kernel decays fast at long times, we model the two bands as a two level system with diagonal terms crossing each other as functions of k, and with an off-diagonal coupling being constant in k. In order to produce the right gap ϵ_g at the crossing point, the off-diagonal coupling should be set to $\epsilon_g/2$. In order to produce the right level spacing, the difference z between the diagonal terms should satisfy $z^2 + \epsilon_g^2 = \epsilon_{10}^2$. Note that, unlike the standard Zener model [10], we do not assume, until later, a linearly varying $z(k)$. The kernel then has the expression (remember $s = at/2$)

$$W = \int_{-\infty}^{\infty} dk\,\frac{z'(k-s)z'(k+s)\epsilon_g^2a^2}{2[\epsilon_{10}(k-s)\epsilon_{10}(k+s)]^2} \times\cos\left[\int_{k-s}^{k+s} dk'\,\epsilon_{10}/a\right], \quad (6)$$

where z' is the derivative of $z(k)$, and we have taken the crossing point (arrow in Fig. 2) as the origin of k. Since the spectrum is symmetric about the crossing point, the stationary phase occurs at $k = 0$, which dominates the integral and yields

$$W = \left[\frac{\pi a}{\epsilon_{10}'(s)}\right]^{1/2}\left[\frac{z'(s)\epsilon_g a}{\epsilon_{10}^2(s)}\right]^2 \times\cos\left[\frac{\pi}{4} + \int_{-s}^{s} dk'\,\epsilon_{10}/a\right], \quad (7)$$

where the prime in $\epsilon_{10}'(s)$ indicates a derivative with respect to the argument. At very large times, we have $\epsilon_{10}(s) \approx s^2/2$, yielding a W that falls off like $t^{-13/2}$.

Next, we examine how the survival probability goes from a quadratic to an exponential decay. Because the kernel has decayed substantially when $\epsilon_{10} \gg \epsilon_g$ according to Eq. (7), we need only to consider the region where ϵ_{10} is comparable to ϵ_g. For this region, we can take $\epsilon_{10}(k) = \sqrt{\epsilon_g^2 + k^2}$, with $z(k) = k$ and $\epsilon_g = V_0$, based on a perturbation theory for the energy gap [9]. Our model reduces to the standard Zener model in this region [10]. The numerical results of Eq. (6) are shown in Fig. 3 (solid lines) for the cases of (a) $\epsilon_g^2/a = 1.6$ and (b) $\epsilon_g^2/a = 5$. One can see that the kernel is an oscillatory decaying function of time, with the frequency of oscillation larger for a larger value of the adiabaticity parameter ϵ_g^2/a. In both cases, the amplitude of the kernel

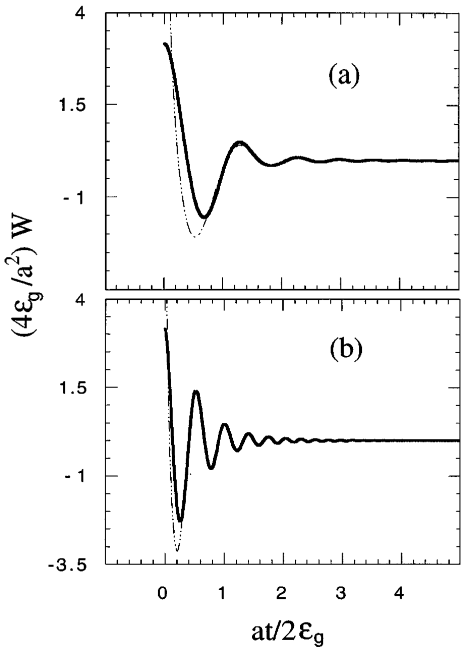

FIG. 3. The scaled transition kernel $(4\epsilon_g/a^2)W$ as a function of the scaled time $at/2\epsilon_g$. The shape of the plot depends only on the adiabaticity parameter ϵ_g^2/a, which is taken as 1.6 in (a) and 5 in (b). The envelopes of both curves fall off in the same fashion, but more coherent oscillations are seen when the adiabaticity parameter is larger. The dash-dotted curves are from the asymptotic expression [Eq. (8)].

has decayed to about half of its initial value $(\pi a^2/4\epsilon_g)$ when $at/2\epsilon_g = 1/2$.

We therefore identify $t_c = \epsilon_g/a$ as the time scale for crossing over from the short-time regime of nonexponential behavior to the long-time regime of exponential decay. Since an exponential decay of the survival probability suggests a loss of coherence, one may also call it the coherence time for our system. We note that the LZ model is valid only when this crossover time is shorter than the Bloch time $t_B = 1/a$, the period of Bloch oscillations. This condition yields $\epsilon_g \ll 1$, which is consistent with our initial assumptions (see Fig. 1).

Our crossover time is also identical in value to a tunneling time of Zener breakdown. In 1986, Büttiker and Landauer [6] considered Zener tunneling of Bloch electrons in an electric field. They chose to consider a tilted band picture, where the electric field is represented by a scalar potential. They found a tunneling time identical to ours, if the electric force is replaced by our inertial force and the lattice constant by π/k_L. The same conclusion can be reached in the vector potential gauge that we used in this Letter, where Zener breakdown becomes LZ transitions between energy dispersions of different Bloch bands. In fact, Mullen *et al.* [7] found the same result in their study of the tunneling time, in the Landauer-Büttiker sense, for LZ transitions by looking at how the transition probability depends on the frequency of a perturbation. It remains to be seen how our crossover time compares with other definitions of tunneling time [14].

The crossover can be seen more clearly from the analytical result of Eq. (7), which takes the following form for the Zener model:

$$W = \frac{(\pi a/s)^{1/2}\epsilon_g^2 a^2}{2[\epsilon_g^2 + s^2]^{7/4}} \cos\left[\frac{\pi}{4} + \int_{-s}^{s} \sqrt{\epsilon_g^2 + k^2}\,\frac{dk}{a}\right]. \tag{8}$$

Since $s = at/2$, the kernel decays with the time scale of ϵ_g/a, the tunneling time discussed above. Within this time, the kernel oscillates with the frequency of the band gap ϵ_g, a very reasonable result. How good is the stationary phase approximation? The dash-dotted lines in Fig. 3 are obtained from the approximation, which agrees with the numerical result almost perfectly after the first minimum. When the adiabaticity parameter ϵ_g^2/a is much larger than unity, such as in the case of Fig. 3(b), the location of the first minimum is estimated to be $t = 3\pi/4\epsilon_g$ using the above expression, which is much smaller than the tunneling time ϵ_g/a. In the case of Fig. 3(a), where the adiabaticity parameter is 1.6, we see from the figure that these two time scales coincide with each other.

Finally, analytic expressions for the short- and long-time characteristic parameters can also be obtained within the Zener model. At very short times, the coefficient of the quadratic time dependence is given by

$$W(0) = \int \frac{a^2\epsilon_g^2\,dk}{2[\epsilon_g^2 + k^2]^2} = \frac{\pi}{4}\frac{a^2}{\epsilon_g}. \tag{9}$$

On the other hand, the rate of exponential decay at long times is found as $\Gamma = (\pi/3)^2 a e^{-(\pi\epsilon_g^2/2a)}$, where we assumed $\epsilon_g^2/a \gg 1$ and used the method of steepest descent. This is a standard result of LZ tunneling except for an unimportant prefactor of $(\pi/3)^2 \approx 1.1$ [15]. The constant term in the long-time exponent of the survival probability is given by

$$\Gamma_0 = -\frac{3\pi}{16}\epsilon_g(a/\epsilon_g^2)^2, \tag{10}$$

which is not exponentially small. A negative value of Γ_0 represents an initial loss of the survival probability over the scale of the tunneling time.

In conclusion, we compared our theory with the data of a recent experiment, and found very good agreement [4]. In the experiment, the band gap was about 0.4, and the acceleration was about 0.1, corresponding to the circle in Fig. 1. The theoretical time dependence of the logarithm of the survival probability for such a situation is plotted in Fig. 4(a). All of the characteristics of the curve, including the initial flat region, the first minimum, and the constant part of the asymptotic exponent Γ_0, agree quantitatively with the experimental data [4].

This experiment cannot yet be used to determine the crossover time or the tunneling time discussed in the main

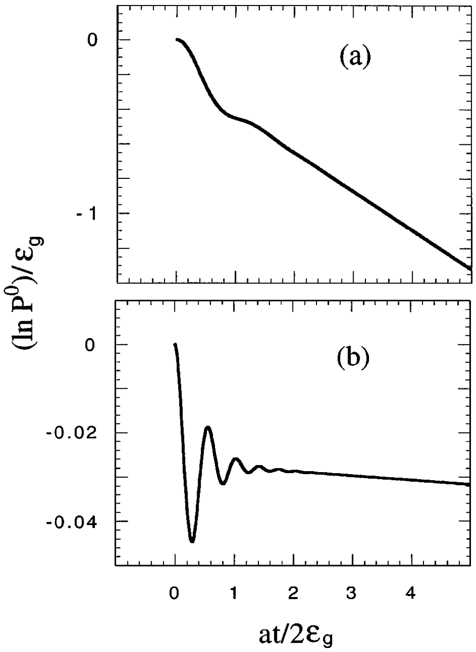

FIG. 4. The scaled exponent of the survival probability $(\ln P_0)/\epsilon_g$ as a function of the scaled time $at/2\epsilon_g$. Again, the shape of the plot depends only on the adiabaticity parameter, which is 1.6 in (a) and 5 in (b). The decay crosses over to an exponential at a time scale of $t = \epsilon_g/a$ in each case. As the tunneling becomes more adiabatic, one sees more oscillations and a smaller drop over the initial stage as well as a smaller asymptotic exponential rate.

text, because all of the time scales of the system become degenerate when $\epsilon_g^2/a \sim 1$. We thus show, in Fig. 4(b), the case of $\epsilon_g^2/a = 5$, where coherent oscillations of the survival probability are prominent. Experimental realization of this regime should address the issue of tunneling time or the crossover time from coherent oscillations to exponential decay. This may be achieved by taking $a = 0.01$ and $\epsilon_g = 0.224$ (square in Fig. 1), well inside the regime of validity of our theory.

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